Symmetric Spaces and Star representations II : Causal Symmetric Spaces

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Abstract

We construct and identify star representations canonically associated with holonomy reducible simple symplectic symmetric spaces. This leads the a non-commutative geometric realization of the correspondence between causal symmetric spaces of Cayley type and Hermitian symmetric spaces of tube type.
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Introduction

The aim of this paper is twofold. We first want to present a generalization of a construction given in [4]. There, a covariant star product (see Definition 1.3) has been defined on a dense open subset of every holonomy reducible simple symplectic (non-Kaehler) symmetric space $M = G/H$, providing a star representation ρ of the transvection algebra $\mathfrak{g}$ of the symmetric space M (see Section 2). Despite the fact that this star product is only defined on an open subset of M, the representation ρ in the case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ exponentiates to the group $G = SL(2,\mathbb{R})$ as a holomorphic discrete series representation. In particular the covariant star product does *not* lead to a representation of $SL(2,\mathbb{R})$ prescribed by the orbit type—indeed, in this case, one has $M = SL(2,\mathbb{R})/SO(1,1)$ which classically yields a principal series representation. It should also be noted that the star representation does

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generally not exponentiate to G as in the case of $G = SL(3, \mathbb{R})$ for instance. In the present work, we prove that the star representation ρ exponentiates to G when the symmetric space $M = G/H$ is of Cayley type. The resulting representation of G turns out to be a holomorphic discrete series representation (when one assigns a particular real value to the deformation parameter); see Theorem 3.4. The proof uses Jordan techniques, providing close relations between Jordan algebras theory and covariant star products.
Second, we would like to indicate how the above construction leads to a (non-commutative) realization of the (algebraic) duality existing between Cayley symmetric spaces and Hermitian symmetric spaces of tube type. Below, we make this assertion more precise.
Let G/K be a Hermitian symmetric space of tube type. Then there exists one and only one (up to isomorphism) symmetric space G/H such that

1. H acts reducibly on $T_{eH}(G/H)$,
2. G/H carries a G-invariant symplectic structure.

This fact leads to the well-known "duality" between Hermitian symmetric spaces of tube type and Cayley symmetric spaces [8]. As such, this duality is algebraic in the sense that it is a correspondence between two lists of involutive simple Lie algebras. Another duality defined on Hermitian symmetric spaces is the so called "compact-non-compact" duality. In this case, the correspondence is not only algebraic. Indeed, denoting by U a compact real form of $G^{\mathbb{C}}$, there is a holomorphic G-equivariant embedding $G/K \to U/K$ underlying the duality. Equivalently, one has a homomorphism of algebras $C^{\infty}(U/K) \to C^{\infty}(G/K)$. Back to Cayley symmetric spaces, our construction of a covariant star product described above leads to a deformation of the (infinitesimal) action of (the Lie algebra) of G on a open subset of G/H. This deformed action turns out to be equivalent to the action of G on the tube domain G/K. Analogously with the compact-non-compact duality, one therefore gets a geometric realization of the correspondence between Cayley symmetric spaces and Hermitian symmetric spaces of tube type.

This paper is organized as follows. In Sections 1 and 2, we recall the notion of covariant star product [1] and results of [4]. Section 3 contains our main result, it starts with recalling some Jordan algebras theory.

1 Covariant $\star$-products

In this section, (M,ω) is a symplectic manifold and $\mathfrak{g}$ is a finite dimensional real Lie algebra. One assumes one has a representation of $\mathfrak{g}$ as an algebra of symplectic vector fields on (M,ω). That is, one has a Lie algebra homomorphism $\mathfrak{g} \to \mathcal{X}(M) : \; X \mapsto X^*$ ($\mathcal{X}(M)$ stands for the space of smooth sections of $T(M)$) such that for all X in $\mathfrak{g}$ one has

$$\mathcal{L}_{X^*}\omega = 0,$$

where $\mathcal{L}$ denotes the Lie derivative. One supposes furthermore that this representation of $\mathfrak{g}$ is strongly Hamiltonian which means that there exists a $\mathbb{R}$-linear map

$$\mathfrak{g} \stackrel{\lambda}{\mapsto} C^\infty(M) : \;\; X \mapsto \lambda_X,$$

such that

$$\begin{array}{ll} (i) & d\lambda_X = i_{X^*}\omega, \; \forall X \in \mathfrak{g}; \\ (ii) & \lambda_{[X,Y]} = \{\lambda_X, \lambda_Y\}, \end{array}$$

where $\{\, , \}$ denotes the Poisson structure on $C^\infty(M)$ associated to the symplectic form ω.

Definition 1.1 *A quadruple $(M,\omega,\mathfrak{g},\lambda)$ with $(M,\omega),\mathfrak{g}$ and λ as above is called a strongly hamiltonian system. The map $\lambda : \mathfrak{g} \mapsto C^\infty(M)$ is called the moment mapping.*

Example 1. Coadjoint orbits. Let $M = \mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit of a Lie group G with Lie algebra $\mathfrak{g}$. In this case, we denote by $\mathfrak{g} \to \mathcal{X}(M) : \; X \mapsto X^*$ the rule which associates to an element X in $\mathfrak{g}$ its *fundamental vector field* on $\mathcal{O}$:

$$X^*_x := \frac{d}{dt}|_0 \, \mathrm{Ad}^*(\exp(-tX))x,$$

where $\mathrm{Ad}^*(g)x$ denotes the coadjoint action of the element $g \in G$ on $x \in \mathfrak{g}^*$.

The formula $\omega^{\mathcal{O}}_x(X^*,Y^*) := \langle x,[X,Y]\rangle$ (with $X,Y \in \mathfrak{g}$) then defines a symplectic structure called after, Kirillov, Kostant and Sauriau, the *KKS symplectic form* on $\mathcal{O}$. Defining, for all $X \in \mathfrak{g}$, the function $\lambda_X \in C^\infty(\mathcal{O})$ by

$$\lambda_X(x) := \langle x, X\rangle,$$

one then has that the quadruple $(\mathcal{O},\omega^{\mathcal{O}},\mathfrak{g},\lambda)$ is a strongly Hamiltonian system.

In the setting of deformation quantization there is a natural way to define the quantization of a classical hamiltonian system [1],[3],[9]. We first recall what deformation quantization (star product) is.

Definition 1.2 *Let $(M, \{\ ,\ \})$ be a Poisson manifold. A star product on $(M, \{\ ,\ \})$ is an associative multiplication $\star_\nu$ on the space $C^\infty(M)[[\nu]]$ of formal power series in the parameter ν with coefficients in the smooth complex-valued functions on M. One furthermore requires the following properties to be true.*

(i) The map $\star_\nu : C^\infty(M)[[\nu]] \times C^\infty(M)[[\nu]] \to C^\infty(M)[[\nu]]$ is $\mathbb{C}[[\nu]]$-bilinear and for all $u \in C^\infty(M) \subset C^\infty(M)[[\nu]]$ one has $u \star_\nu 1 = 1 \star_\nu u = u$; ($\mathbb{C}[[\nu]]$ denotes the field of power series in ν with (constant) complex coefficients).

(ii) For all $u, v \in C^\infty(M)$ one has,

$$\begin{array}{ll} 1. & u \star_\nu v \,\mathrm{mod}(\nu) = uv, \\ 2. & (u \star_\nu v - v \star_\nu u) \,\mathrm{mod}(\nu^2) = 2\nu\{u, v\}. \end{array}$$

In other words, a star product is an associative formal deformation of the pointwise multiplication of functions in the direction of the Poisson structure.

Example 2. The Moyal star product on $\mathbb{R}^{2n}$. We fix $M = \mathbb{R}^{2n}$ (or an open set in $\mathbb{R}^{2n}$) and $\omega = \sum_{i<j} \Lambda_{ij} dx^i \wedge dx^j$ with constant coefficients Λ_{ij}'s.

Let $u, v \in C^\infty(\mathbb{R}^{2n})$ and define their Moyal product by the formal power series:

$$u \star_\nu^0 v := uv + \sum_{k=1}^{\infty} \frac{\nu^k}{k!} \Lambda^{i_1 j_1} \dots \Lambda^{i_k j_k} \frac{\partial^k}{\partial x^{i_1} \dots \partial x^{i_k}} u \, \frac{\partial^k}{\partial x^{j_1} \dots \partial x^{j_k}} v.$$

The $\mathbb{C}[[\nu]]$-bilinear extension of the above product to $C^\infty(\mathbb{R}^{2n})[[\nu]]$ then defines a star product on $(\mathbb{R}^{2n}, \Lambda)$ called Moyal star product.

Definition 1.3 *[1] Let $(M, \omega, \mathfrak{g}, \lambda)$ be a classical strongly hamiltonian system. A star product $\star_\nu$ on (M, ω) is called $\mathfrak{g}$-covariant if for all $X, Y \in \mathfrak{g}$ one has*

$$\lambda_X \star_\nu \lambda_Y - \lambda_Y \star_\nu \lambda_X = 2\nu\{\lambda_X, \lambda_Y\}.$$

In order to avoid technical difficulties in defining the star representation (see below), we will assume our covariant star products to satisfy the following condition.

Definition 1.4 *Let $(M, \omega, \mathfrak{g}, \lambda)$ be a strongly hamiltonian system. Let $\star_\nu$ be a $\mathfrak{g}-$covariant star product on (M, ω). We say that $\star_\nu$ has the property (B) if there exists an integer $N \in \mathbb{N}$ such that one has*

$$\begin{array}{l} (\lambda_X \star_\nu u) \,\mathrm{mod}(\nu^N) = (\lambda_X \star_\nu u) \,\mathrm{mod}(\nu^{N+n}); \\ (u \star_\nu \lambda_X) \,\mathrm{mod}(\nu^N) = (u \star_\nu \lambda_X) \,\mathrm{mod}(\nu^{N+n}), \end{array}$$

for all $X \in \mathfrak{g}$ and $u \in C^\infty(M)$ and $n \in \mathbb{N}$.

In other words the series $\lambda_X \star_\nu u$ and $u \star_\nu \lambda_X$ stop at order N independently of u in $C^\infty(M)$.

The data of a covariant star product satisfying the property (B) yields representations of $\mathfrak{g}$. Let $E_\nu := C^\infty(M)[[\nu, \frac{1}{\nu}]]$ be the space of formal power series in ν and $\frac{1}{\nu}$ with coefficients in $C^\infty(M)$. Assume that the $\mathfrak{g}$-covariant star product $\star_\nu$ on (M,ω) satisfies the property (B). Then for all $X \in \mathfrak{g}$ and $a = \sum_{\ell\in\mathbb{Z}} \nu^\ell a_\ell \in E_\nu$, the expression

$$\lambda_X \star_\nu a := \sum_{\ell\in\mathbb{Z}} \nu^\ell (\lambda_X \star_\nu a_\ell),$$

defines an element of E_ν. Indeed, let $c_k : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ be the k^{th} cochain of $\star_\nu$, that is

$$u \star_\nu v =: \sum \nu^k c_k(u,v), \;\; u, v \in C^\infty(M).$$

Then,

$$\lambda_X \star_\nu a = \sum_{\ell\in\mathbb{Z}} \nu^\ell \sum_{k\leq N} \nu^k c_k(\lambda_X, a_\ell) = \sum_{m\in\mathbb{Z}} \nu^m \left(\sum_{k+\ell=m} c_k(\lambda_X, a_\ell) \right).$$

Therefore, each sum occurring in the parentheses has only a finite number of terms since $0 \leq k \leq N$.

Definition 1.5 *Let $(M,\omega,\mathfrak{g},\lambda)$ be a strongly hamiltonian system and $\star_\nu$ be a $\mathfrak{g}$-covariant star product on (M,ω) satisfying the property (B). One defines the representations ρ^L and ρ^R of $\mathfrak{g}$ on E_ν by*

$$\begin{aligned} \rho^L(X)a := & \quad \frac{1}{2\nu}(\lambda_X \star_\nu a) \text{ and} \\ \rho^R(X)a := & \quad \frac{1}{2\nu}(a \star_\nu \lambda_X). \end{aligned}$$

Definition 1.6 *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $(M,\omega,\mathfrak{g},\lambda)$ be a strongly hamiltonian system. Let $\star_\nu$ be a $\mathfrak{g}$-covariant star product on (M,ω) satisfying the property (B). The associated star representation of G (if it exists) is the representation π^L of G on E_ν such that*

$$d\pi^L = \rho^L.$$

2 Holonomy reducible symplectic symmetric spaces

Let G be a connected simple Lie group. Let us denote by $\mathfrak{g}$ its Lie algebra. Let $\mathcal{O}$ be an adjoint orbit of G in $\mathfrak{g}$. Choose a base point o

in $\mathcal{O}$ and denote by $\mathfrak{h}$ the Lie algebra of its stabilizer in $\mathfrak{g}$. Since $\mathfrak{g}$ is simple the *Killing form* β establishes an equivariant linear isomorphism between $\mathfrak{g}$ and its dual $\mathfrak{g}^\star$. Therefore, every adjoint orbit can be identified with a coadjoint one. We will denote by $\omega^{\mathcal{O}}$ the KKS symplectic structure on $\mathcal{O}$ (cf. Example 1).

Definition 2.1 *An adjoint orbit $\mathcal{O}$ in $\mathfrak{g}$ is symmetric if there exists an involutive automorphism σ of $\mathfrak{g}$ such that $\mathfrak{h} = \{X \in \mathfrak{g} | \sigma(X) = X\}$. In this case, we denote by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the decomposition of $\mathfrak{g}$ induced by σ ($\sigma|_{\mathfrak{q}} = -id_{\mathfrak{q}}$).*

One has $[\mathfrak{h}, \mathfrak{q}] = \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{h}$.
In a symmetric situation, the KKS form induces on the vector space $\mathfrak{q}$ a bilinear symplectic form which we denote by Ω; this form is invariant under the action of $\mathfrak{h}$. The triple $(\mathfrak{g}, \sigma, \Omega)$ is called a *simple symplectic symmetric triple* [5]. Symmetric orbits have been studied in [2]. In particular, on has.

Proposition 2.1 *Let $\mathcal{O}$ be a symmetric adjoint orbit of a simple Lie group G and $(\mathfrak{g}, \sigma, \Omega)$ be its associated symplectic symmetric triple. The following assumptions are equivalent.*

- *(i) the center $\mathfrak{z}(\mathfrak{h})$ of $\mathfrak{h}$ contains a non-compact element.*
- *(ii) The subspace $\mathfrak{q}$ splits into a direct sum $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{l}'$ of isomorphic $\mathfrak{h}$-modules. One has $[\mathfrak{l}, \mathfrak{l}] = 0\,, [\mathfrak{l}', \mathfrak{l}'] = 0$ and both $\mathfrak{l}$ and $\mathfrak{l}'$ are β-isotropic and Ω-Lagrangian subspaces of $\mathfrak{q}$.*

Such a symmetric orbit is called *holonomy reducible* if $\mathfrak{h}$ acts reducibly on $\mathfrak{q}$.

Proposition 2.2 *Let $\mathcal{O}$ be a holonomy reducible symmetric orbit in $\mathfrak{g}$. We define the map $\phi : \mathfrak{q} = \mathfrak{l} \oplus \mathfrak{l}' \to \mathcal{O}$ by*

$$\phi(l, l') := Ad(exp(l).exp(l')).o$$

Then ϕ is a Darboux chart on $(\mathcal{O}, \omega^{\mathcal{O}})$. Precisely, one has $\phi^\star \omega^{\mathcal{O}} = \Omega$.

We transport the infinitesimal action of $\mathfrak{g}$ on $\phi(\mathfrak{q}) \subset \mathcal{O}$ to an infinitesimal action of $\mathfrak{g}$ on $\mathfrak{q}$ via ϕ. One then gets a homomorphism of Lie algebra

$$\mathfrak{g} \to \chi(\mathfrak{q}) : X \to X^\star.$$

Setting

$$\lambda_A(x) := \beta(\phi(x), A) \qquad (x \in \mathfrak{q}, A \in \mathfrak{g}),$$

one obtains the strongly hamiltonian system (cf.Definition 1.1) $(\mathfrak{q}, \Omega, \mathfrak{g}, \lambda)$. The main property of the Darboux chart ϕ is

Proposition 2.3 *The Moyal star product on the symplectic vector space $(\mathfrak{q}, \Omega)$ is $\mathfrak{g}$-covariant. Moreover the Moyal star product in this case satisfies the property (B).*

Definition 2.2 *Let $\mathcal{S}'_2(\mathfrak{q})$ be the space of distributions on $\mathfrak{q}$ which are tempered in the $\mathfrak{l}'$-variables w.r.t. the Lebesgue measure dl' on $\mathfrak{l}'$. On $\mathcal{S}'_2(\mathfrak{q})$, we consider the partial Fourier transform $\mathcal{F} : \mathcal{S}'_2(\mathfrak{q}) \to \mathcal{S}'_2(\tilde{\mathfrak{q}})$ which reads formally as*

$$(\mathcal{F}u)(l, \eta) := \int_{\mathfrak{l}'} e^{-i\Omega(\eta, l')} u(l, l') dl' \qquad i := \sqrt{-1}$$

Here $\tilde{\mathfrak{q}} := \mathfrak{l} \oplus \mathfrak{l}$ i.e. we identify the dual space $\mathfrak{l}'^{\star}$ with $\mathfrak{l}$ by use of Ω. We will also adopt the notation $\hat{u} := \mathcal{F}u$

We define the $\mathbb{R}$-isomorphism

$$\tilde{\mathfrak{q}} \to \mathfrak{l}^{\mathbb{C}} : (l, \eta) \to z = l + i\nu\eta$$

where the parameter ν is now considered as being *real*.
In $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{l}'$, we choose an Ω-symplectic basis $\{L_a, L'_a;\ 1 \le a \le n\}$ where $L_a \in \mathfrak{l}$ and $L'_a \in \mathfrak{l}'$. On $\mathfrak{l}$, we define the coordinate system $x = \Omega(x, L'_a)L_a =: x^a L_a$.
If h is an element of $\mathfrak{h}$, we define its trace as

$$\mathrm{spur}(h) := \Omega([h, L_a], L'_a).$$

In this setting, one has the holomorphic constant vector field on $\mathfrak{l}^{\mathbb{C}}$:

$$\partial_{z^a} := \frac{1}{2\nu}(\nu(L_a)_l - i(L_a)_\eta) \quad (1 \le a \le n).$$

Definition 2.3 *Considering $\mathfrak{l}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$, we define, for all $A \in \mathfrak{g}^{\mathbb{C}}$, the polynomials on $\mathfrak{l}^{\mathbb{C}}$:*

$$\begin{aligned} h_A^{\mathbb{C}}(z) &:= A_{\mathfrak{h}^{\mathbb{C}}} + [A_{\mathfrak{l}'^{\mathbb{C}}}, z] \in \mathfrak{h}^{\mathbb{C}} \\ l_A^{\mathbb{C}}(z) &:= A_{\mathfrak{l}^{\mathbb{C}}} + [A_{\mathfrak{h}^{\mathbb{C}}}, z] + \tfrac{1}{2}[z, [z, A_{\mathfrak{l}'^{\mathbb{C}}}]] \in \mathfrak{l}^{\mathbb{C}} \end{aligned}$$

where $z \in \mathfrak{l}^{\mathbb{C}}$ and $A = A_{\mathfrak{h}^{\mathbb{C}}} + A_{\mathfrak{l}^{\mathbb{C}}} + A_{\mathfrak{l}'^{\mathbb{C}}}$ according to the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{l}'^{\mathbb{C}}$.

Definition 2.4 *For all $A \in \mathfrak{g}^{\mathbb{C}}$, we define the holomorphic vector field $\mathcal{Z}_A^{(\nu)} \in \Gamma(T^{1,0}(\mathfrak{l}^{\mathbb{C}}))$ by*

$$(\mathcal{Z}_A^{(\nu)})_z.f := (l_A^{\mathbb{C}}(z))^a(\partial_{z^a}.f)(z)$$

where $z \in \mathfrak{l}^{\mathbb{C}}$, $f \in C^\infty(\mathfrak{l}^{\mathbb{C}}, \mathbb{C})$ and where, for all $w = w_1 + iw_2 \in \mathfrak{l}^{\mathbb{C}} = \mathfrak{l} \oplus i\mathfrak{l}$, we set $w^a := w_1^a + iw_2^a \qquad (1 \leq a \leq n)$.
In the same way, extending the Killing form β and the trace spur $\mathbb{C}$-linearly to $\mathfrak{h}^{\mathbb{C}}$, we define the complex polynomial of degree 1 :

$$\tau_A^{(\nu)} := \frac{1}{2\nu}(\beta(h_A^{\mathbb{C}}, o) + \nu\, spur(h_A^{\mathbb{C}})).$$

Proposition 2.4 *For all $A \in \mathfrak{g}$, one has*

$$\frac{1}{2\nu}\mathcal{F}(\lambda_A \star_\nu u) = \tau_A^{(\nu)}.\hat{u} + \mathcal{Z}_A^{(\nu)}.\hat{u}$$

where u is chosen in such a way that the LHS and the RHS make sense (e.g. $u \in \mathcal{S}(\mathfrak{q})$ the Schwartz space on $\mathfrak{q}$).

In particular, the formula

$$\hat{\rho}(A)f := \tau_A^{(\nu)}f + \mathcal{Z}_A^{(\nu)}.f \qquad (A \in \mathfrak{g}, f \in C^\infty(\mathfrak{l}^{\mathbb{C}})[[\nu, \frac{1}{\nu}]]) \qquad (1)$$

defines a (holomorphic) representation of $\mathfrak{g}$ on $E_\nu := C^\infty(\mathfrak{l}^{\mathbb{C}})[[\nu, \frac{1}{\nu}]])$.

3 Main Theorem

3.1 Euclidean Jordan algebras and Tube Domains

An algebra V over $\mathbb{R}$ or $\mathbb{C}$ is said to be a Jordan algebra if for all elements x and y in V, one has

$$\begin{aligned} x \cdot y &= y \cdot x, \\ x \cdot (x^2 \cdot y) &= x^2 \cdot (x \cdot y). \end{aligned}$$

For an element $x \in V$ let $L(x)$ be the linear map of V defined by $L(x)y := x \cdot y$. We denote by $\tau(x, y)$ the symmetric bilinear form on V defined by $\tau(x, y) = \operatorname{Tr} L(x \cdot y)$.

A Jordan algebra V is semi-simple if the form τ is non degenerate on V. A semi-simple Jordan algebra is unital, we denote by e its identity element.

One defines on V the triple product

$$\{x, y, z\} := (x \cdot y) \cdot z + x \cdot (y \cdot z) - y \cdot (x \cdot z).$$

We denote by $x \,\square\, y$ the endomorphism of V defined by $x \,\square\, y(z) := \{x, y, z\}$. Remark that $x \,\square\, y = L(x \cdot y) + [L(x), L(y)]$ See [11] for more details.

Definition 3.1 *A Jordan algebra V_o over $\mathbb{R}$ is said to be Euclidean if the bilinear form $\tau(x, y)$ is positive definite on V_o.*

Let V_o be an Euclidean Jordan algebra (EJA), then the set

$$C := \{x^2 \mid x \text{ invertible in } V_o\}$$

is an open, convex, self-dual cone in V_o. Those properties of C actually characterize V_o as an EJA.
Let V be the complexification of V_o. Consider the tube $T_C = V_o + iC \subset V$ and the Lie group $Aut(T_C)$ of holomorphic automorphisms of T_C. We denote by $G = (Aut(T_C))_o$ its identity connected component.

It can be shown that every element X of the Lie algebra $\mathfrak{g}$ of the group $Aut(T_C)$ is a holomorphic vector field on the tube T_C of the form (see [7] p.209)

$$X(z) = u + Tz + P(z)v,$$

where $u, v \in V_o$, T is a linear map of the form $T = a \,\square\, b$ with $a, b \in V_o$ and $P(z) = 2L(z)^2 - L(z^2)$.

In other words, the Lie algebra $\mathfrak{g}$ is a symmetric Lie algebra admitting the following graduation

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_{-1} := V_o$ is the set of constant polynomial vector fields on T_C acting by translations, $\mathfrak{g}_0 := V_o \,\square\, V_o$ is the subset of $\mathfrak{gl}(V_o)$ preserving the cone C and $\mathfrak{g}_1 := \{P(z)v = \{z, v, z\} \mid v \in V_o\}$ is a subset of homogeneous polynomial maps $V \mapsto V$ of degree 2. Remark that $\mathfrak{g}_1 \simeq \mathrm{Ad}(j)V_o$ where $j \in G$ denotes the Jordan inverse $j(z) = -z^{-1}$.

One writes (u, T, v) for $X \in \mathfrak{g}$. The following result is classical.

Proposition 3.2 *For $X = (u, T, v)$ and $X' = (u', T', v')$ in $\mathfrak{g}$ one has*

$$[X, X'] = (Tu' - T'u,\ 2u' \,\square\, v + [T, T'] - 2u \,\square\, v',\ T'^{\sharp}v - T^{\sharp}v'),$$

where $T^{\sharp}$ denotes the adjoint endomorphism with respect to τ.
The map $\theta : (u, T, v) \mapsto (v, -T^{\sharp}, u)$ is an involutive automorphism of $\mathfrak{g}$, such that $\theta(\mathfrak{g}_i) = \mathfrak{g}_{-i}$, $i \in \{-1, 0, 1\}$. The Lie algebra $\mathfrak{g}$ is semi-simple.
The Killing form on $\mathfrak{g}$ is given by

$$\beta(X, X') = \beta_o(T, T') + 2\,\mathrm{tr}(TT') - 4\tau(u, v') - 4\tau(v, u'),$$

where β_o denotes the standard Killing form on $\mathfrak{gl}(V_o)$.

Note that using this identification we have, for $x, y, z \in V_o = \mathfrak{g}_{-1}$

$$x \,\square\, y = -\frac{1}{2}[x, \theta y] \quad \text{and} \tag{2}$$

$$\{x, y, z\} = -\frac{1}{2}\,[[x, \theta y], z]\,. \tag{3}$$

The EJA's and their corresponding symmetric Lie algebras are given by the following table.

$\mathfrak{g}$	$\mathfrak{k}$	V	V_o
$\mathfrak{su}(n,n)$	$\mathfrak{su}(n) \oplus i\mathbb{R}$	$M(n,\mathbb{C})$	$Herm(n,\mathbb{C})$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{su}(n) \oplus i\mathbb{R}$	$Sym(n,\mathbb{C})$	$Sym(n,\mathbb{R})$
$\mathfrak{so}^*(4n)$	$\mathfrak{su}(2n) \oplus i\mathbb{R}$	$Skew(2n,\mathbb{C})$	$Herm(n,\mathbb{H})$
$\mathfrak{so}(n,2)$	$\mathfrak{so}(n) \oplus i\mathbb{R}$	$\mathbb{C}^n$	$\mathbb{R}^n$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6 \oplus i\mathbb{R}$	$Herm(3,\mathbb{O}) \otimes \mathbb{C}$	$Herm(3,\mathbb{O})$

Adopting to our setting results of [8], one gets

Proposition 3.3 *(i) The group $G = Aut(T_C)_o$ admits a symmetric holonomy reducible coadjoint orbit.*
(ii) Let K be the maximal compact subgroup of G. Then the symmetric space $G/K \simeq T_C$ is an Hermitian symmetric space of tube type.

3.2 Holomorphic discrete Series

Let $G = (\mathrm{Aut}(T_C))_o$ be the Hermitian Lie group associated with an Euclidean Jordan algebra V_o. We denote by n the dimension of V_o and by r its rank. On V_o one defines (in a canonical way) two homogeneous polynomials $\Delta(x)$ of degree r and $\mathrm{tr}(x)$ of degree 1 which coincide with usual determinant and trace in case of matrix Jordan algebras. (see [7] for more details).

For a real parameter m consider the space $H^2_m(T_C)$ of holomorphic functions $f \in \mathcal{O}(T_C)$ such that

$$\|f\|^2_m = \int_{T_C} |f(z)|^2 \Delta^{m-2\frac{n}{r}}(y)dxdy < \infty,$$

where $z = x + iy \in T_C$. Note that the measure $\Delta^{-2\frac{n}{r}}(y)dxdy$ on T_C is invariant under the action of the group G. For m in the Wallach set $\mathcal{W}$ (see [7] p.264) these spaces are non empty Hilbert spaces with reproducing kernels.

The action of G on $H^2_m(T_C)$ $(m \in \mathcal{W})$ given by

$$\pi_m(g)f(z) = \mathrm{Det}^m(D_{g^{-1}}(z))f(g^{-1}.z) \tag{4}$$

is called a *holomorphic discrete series representation.*

In the above formula by $D_{g^{-1}}(z)$ denotes the derivate map of the conformal transformation $z \to g^{-1}.z$ of the tube.

The derivate representation

$$d\pi_m(X)\phi(z) = -m\frac{r}{n}\operatorname{Tr} DX(z)\cdot\phi(z) - D\phi(z)(X(z)). \qquad (5)$$

is then given by the following formulæ([10]).

$$\begin{array}{ll} \text{for } X(z)=(u,0,0) & d\pi_m(X)\phi(z) = -D_u\phi(z) \\ \text{for } X(z)=(0,0,v) & d\pi_m(X)\phi(z) = -2m\tau(z,v)\phi(z) - D_{p_v(z)}\phi(z) \\ \text{for } X(z)=(0,T,0) & d\pi_m(X)\phi(z) = -m\dfrac{r}{n}\operatorname{Tr} T\phi(z) - D_{T(z)}\phi(z), \end{array}$$

where $D_A\phi(z)$ denotes the action of the tangent vector A on the function ϕ at the point $z \in V$.

Theorem 3.4 *The star representation ρ^L_ν associated to $\star_\nu$ is equivalent to the derivate holomorphic discrete series representation $d\pi_m$ for*

$$m = \frac{\beta(o,o) + n\nu c}{4\nu r c}$$

where c is the eigenvalue of the adjoint action of the base point o.

Proof. We use the notations of Section 2. Using the identification formulæ (2) and (3), one gets that

$$\mathcal{Z}^{(\nu)}_X = X(z).$$

Let us now discuss the expression for $\tau^{(\nu)}_X = \frac{1}{2\nu}(\beta(h^{\mathbb{C}}_X, o) + \nu\operatorname{spur}(h^{\mathbb{C}}_X))$.

The Lie algebra $\mathfrak{h}$ is reductive and therefore it admits the following decomposition $\mathfrak{h} = \mathfrak{z}(h) \oplus [\mathfrak{h},\mathfrak{h}]$, where $\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$. We write $H = H_z + H_d$ according to this decomposition. Because the trace function vanishes on the Lie algebra commutator we have $\operatorname{spur}(H) = \operatorname{spur}(H_z)$. But the center of $\mathfrak{h}$ is one dimensional (see 2.1) and any element $H_z \in \mathfrak{z}(\mathfrak{h})$ can be written as $H_z = h_z \cdot o$. Therefore $\tau^{(\nu)}_H = \frac{1}{2\nu}[\beta(H_z, o) + \nu\operatorname{spur}(H_z)]$. Furthermore we have $\beta(H_z, o) = h_z\beta(o,o)$ and

$$\operatorname{spur}(H_z) = h_z \cdot \operatorname{ad}(o)|_{\mathfrak{l}} = h_z c \cdot \operatorname{id}|_{\mathfrak{l}} = h_z n c.$$

In other words it means that τ^ν_X is proportional to $\operatorname{Tr}(\operatorname{ad}(h^{\mathbb{C}}_X)|_{\mathfrak{l}})$.

Observe now that for the polynomial $h^{\mathbb{C}}_X(z)$ introduced in Definition 2.3 one has $h^{\mathbb{C}}_X(z) = -2DX(z)$. So, finally we have

$$\tau^{(\nu)}_X = -\frac{\beta(o,o) + n\nu c}{4n\nu c}\operatorname{Tr} DX.$$

The identification of corresponding terms in formulæ (1) and (5) completes the proof. $\square$

Remark. The vector field part $\mathcal{Z}_X^{(\nu)}$ of the star representation $\hat{\rho}(X)$ $X \in \mathfrak{g}$ (see (1)) is in general singular on the entire $V = \mathfrak{l}^{\mathbb{C}}$. By this we mean that, denoting by ϕ_t^X the local flow of X, the set of "bad" Cauchy datas $S_X := \{z \in V \mid \phi_t^X(z)$ is not defined for all values of $t\}$ is in general not empty. However, in the case where the group G is the automorphism group of a tube domain, Theorem 3.4 shows that the complementary set U of $\cup_{X\in\mathfrak{g}} S_X = V_o$ in V is not empty. The infinitesimal action of $\mathfrak{g}$ on (a connected component of) U exponentiates to G as its action on the holomorphic tubular realization of G/K. A deep geometric study and in particular relations between star representations and Riccati type ordinary differential equations (see [12]) has not been investigated here. However, one can at least say that, given a holonomy reducible symmetric orbit $\mathcal{O}$ of G, one then canonically gets a one parameter family of representations $\{\rho_\nu\}_{\nu\in\mathbb{R}}$ of $\mathfrak{g}$ deforming the infinitesimal action (ρ_0) of $\mathfrak{g}$ on an open subset $\phi(\mathfrak{p})$ of $\mathcal{O}$. This parameter family leads to an interpolation between the infinitesimal action of $\mathfrak{g}$ on G/H and its holomorphic action on G/K (ρ_ν for the value $-\frac{1}{nc}\beta(o,o)$ of ν).

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